

AD-A072 622

FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS  
ACCELERATED LIFE TESTING - A PRAGMATIC BAYESIAN APPROACH.(U)  
JUL 78 F PROSCHAN, N D SINGPURWALLA

F/G 12/1

N00014-77-C-0263

UNCLASSIFIED

FSU-STATISTICS-M474

AFOSR-TR-78-1427

NL

| OF |

AD  
A072 622



END  
DATE  
FILMED

9-79

DDC



AFOSR-TR- 78-1427

LEVEL

2

AD A 072622

ACCELERATED LIFE TESTING -  
A PRAGMATIC BAYESIAN APPROACH

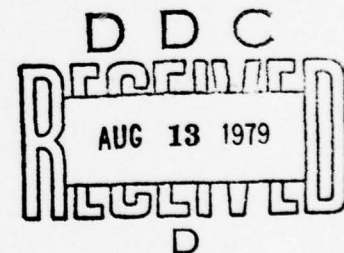
by

Frank Proschan\*  
The Florida State University  
and

Nozer D. Singpurwalla\*\*  
The George Washington University

FSU Statistics Report M474  
AFOSR Technical Report No. 80

July, 1978  
The Florida State University  
Department of Statistics  
Tallahassee, Florida 32306



DDC FILE COPY

- \* Research supported by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR-74-2581.
- \*\* Research supported by the Office of Naval Research Contract N00014-77-C-0263, Project NR 042-372.

Reproduction in whole or in part is permitted for any purpose of the United States Government.

Approved for public release;  
distribution unlimited.

AIR FORCE OFFICE OF SCIENTIFIC RESEARCH (AFSC)  
NOTICE OF TRANSMITTAL TO DDC  
This technical report has been reviewed and is  
approved for public release IAW AFR 190-12 (7b).  
Distribution is unlimited.  
A. D. BLOSE  
Technical Information Officer

A. D. BLOSE  
Technical Information Officer



REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER (18) AFOSR-TR-78-1427	2. GOVT ACCESSION NO. (19) TR-78-1427	3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) (6) ACCELERATED LIFE TESTING - A PRAGMATIC BAYESIAN APPROACH	5. TYPE OF REPORT & PERIOD COVERED (9) Interim Repts	6. PERFORMING ORG. REPORT NUMBER FSU No. M474	
7. AUTHOR(s) (10) Frank/Proschan Nozer D./Singpurwalla	8. CONTRACT OR GRANT NUMBER(s) (15) N00014-77-C-0263 ✓ AFOSR-74-2581	9. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (16) 61102F (17) 2304/A5 (17) A5	
10. PERFORMING ORGANIZATION NAME AND ADDRESS The Florida State University Department of Statistics Tallahassee, Florida 32306	11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332	12. REPORT DATE (11) July 1978	13. NUMBER OF PAGES 20
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (12) 24 p1	15. SECURITY CLASS. (of this report) UNCLASSIFIED	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report)  Approved for public release ; distribution unlimited.			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) (14) FSU-STATISTICS-M474, TR-80-AFOSR			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Accelerated life testing      Pooling of adjacent violators Non-parametric Bayesian approach      Acceleration Function Ordered failure rates      Time transformation function Isotonized estimates      Overstress life tests			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) In this paper we propose a model for accelerated life testing which differs from those considered in the past. Our model does not require the usual assumptions about the failure distributions and the acceleration functions. Our approach is Bayesian, and depends on the estimation of ordered failure rate functions at different stress levels. Since our model is nonparametric, and our approach is quite pragmatic, our results should be of use to engineers working on accelerated life tests, and biometricians involved in carcinogenic experiments.			

ACCELERATED LIFE TESTING -  
A PRAGMATIC BAYESIAN APPROACH

by

Frank Proschan  
Nozer D. Singpurwalla

ABSTRACT

In this paper we propose a model for accelerated life testing which differs from those considered in the past. Our model does not require the usual assumptions about the failure distributions and the *acceleration functions*. Our approach is Bayesian, and depends on the estimation of ordered failure rate functions at different stress levels. Since our model is nonparametric, and our approach is quite pragmatic, our results should be of use to engineers working on accelerated life tests, and biometricians involved in carcinogenic experiments.

Accession For	
NTIS GRA&I	<input checked="checked" type="checkbox"/>
DDC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
Justification	
By	
Distribution/	
Availability Codes	
Dist.	Availand/or special
A	

## 0. Introduction and Overview

In practice, long life items are often subjected to larger than normal stresses (doses) in order to obtain failure data in a short amount of test time. It is common to test the items at more than one stress level, and typically, more items are tested at the higher stress levels than those at the lower stress levels; such tests are known as *accelerated or overstress* life tests. The basic aim is to make inferences about the life distribution of the items at the normal stress levels using the failure data from accelerated tests.

The current approach to this important problem makes inferences under parametric assumptions. This may be valid in some situations, and does yield results which are appealing from a statistical point of view. However, an engineer or a statistician working under less well-defined conditions may find this approach too restrictive. For example, it is common to assume that at all stress levels, failure times are governed by exponential distributions or Weibull distributions. In addition, a functional relationship between the parameters of the failure distribution and the applied stress is assumed. Such a relationship is known as an *acceleration function* or a *time transformation function*; examples can be found in Mann, Schafer, and Singpurwalla (Chapter 9, (1974)). In two recent papers, Phatak, Zimmer, and Williams (1977) and Shaked, Zimmer and Ball (1977), the distributional assumption is dropped, but the requirement is retained that the (unknown) failure distribution be of the same form at the use and at all the accelerated stress levels.



Clearly, there are many situations where the above assumptions may not be appropriate. Of particular concern is the assumption that the failure distribution is of the same form at all the stress levels. One reason for this concern is that different stress levels may have different effects on the mechanism which causes failure, and thus from a physical point of view, it may be more realistic to allow for different forms of the failure distribution at the different stress levels.

The approach that we propose in this paper requires neither distributional assumptions nor the specification of a time transformation function. Rather, our approach is Bayesian, and is prompted by what is actually done in practice. The Bayesian point of view allows us to incorporate some *a priori* information which is available in accelerated life tests. We would like this paper to be construed as preliminary and pragmatic, and thus have not attempted to give a full theoretical justification. Consequently, we would like to invite mathematical statisticians to resolve the statistical problems posed.

## 1. Preliminaries

As stated before, in accelerated life tests the items are tests under different stress environments. A stress environment may be characterized by a single stress such as voltage or temperature, or by multiple stresses, each of a different type. We denote a stress environment by  $E$ , and the set of all  $E$ 's by  $\mathcal{E}$ . We assume that the elements of  $\mathcal{E}$  may be ordered according to the magnitude of their severity. Thus, for any two elements  $E_i$  and  $E_j$  belonging to  $\mathcal{E}$ ,  $E_i > E_j$  denotes the fact that  $E_i$  is more severe than  $E_j$ . Let us denote the  $k$  accelerated stress environments by

$E_1, E_2, \dots, E_k$ , and the normal or use conditions stress environment by  $E_u$ . We shall assume that the set  $E$  is completely ordered with respect to the relationship  $>$  and that

$$E_1 > E_2 > \dots > E_k > E_u.$$

The basic problem in accelerated life testing is to make inferences about the life distribution of the item under use environment  $E_u$  using failure data obtained under accelerated stress environments  $E_1, E_2, \dots, E_k$ , where  $k \geq 1$ . In some situations, it is possible to obtain a limited amount of failure data under the use conditions environment  $E_u$ ; however, in practice, these situations do not appear to be very common. Although having failure data under the use conditions environment  $E_u$  has advantages, its absence is in no way detrimental to our approach. However, in order to obtain results which are useful, it is necessary that  $k$  be moderate to large, and this is what we require. This requirement does not impose any practical difficulties in many situations of interest, especially those involving the accelerated life testing of electronic components and in bioassay experiments on animal populations.

In order to introduce some notation, let us denote the failure distribution of the items which are tested under environment  $E_j$  by  $F_j$ , where  $F_j(0^-) \equiv 0$  for all values of  $j$ . We assume that  $F_j$  is absolutely continuous and thus  $f_j(x)$ , its probability density function, exists for  $x \in [0, \infty)$ . If we denote  $1 - F_j(x)$  by  $\bar{F}_j(x)$ , then  $\lambda_j(x)$ , the failure rate of  $F_j(x)$ , is defined by

$$\lambda_j(x) = \frac{f_j(x)}{\bar{F}_j(x)}, \quad \bar{F}_j(x) > 0.$$

We find it convenient and reasonable to assume that  $\lambda_j(x)$  is continuous in  $x$  for  $x \in [0, \infty)$ . We use the terms *failure rate* and *hazard rate* interchangeably. Since  $E_1 > E_2 > \dots > E_k > E_u$ , it is logical to assume that

$$\lambda_1(x) \geq \lambda_2(x) \geq \dots \geq \lambda_k(x) \geq \lambda_u(x) \quad (1.1)$$

for all  $x \in [0, \infty)$ .

Using failure data obtained under  $E_1, E_2, \dots, E_k$ , we would like to obtain estimators  $\hat{\lambda}_j(x)$  of  $\lambda_j(x)$ ,  $j = 1, 2, \dots, k$ , such that for some  $0 \leq L < \infty$  and all  $x \in [0, L]$ ,

$$\hat{\lambda}_1(x) \stackrel{st}{\geq} \hat{\lambda}_2(x) \stackrel{st}{\geq} \dots \stackrel{st}{\geq} \hat{\lambda}_k(x). \quad (1.2)$$

The notation  $X \stackrel{st}{\geq} Y$  denotes the fact that  $X$  is stochastically larger than  $Y$ , i.e.,

$$P[X \geq x] \geq P[Y \geq x] \quad \text{for all values of } x.$$

In order to obtain estimators of  $\lambda_j(x)$ ,  $j = 1, 2, \dots, k$ , which satisfy Equation (1.2), we shall use a Bayesian approach. Under this approach, Condition (1.1) is incorporated as a prior assumption. Our approach is in contrast with that of Brunk, Franck, Hanson and Hogg (1966), who embody a similar but weaker condition than (1.1) in their likelihood function.

Specifically, Brunk et al. assume that the distributions are ordered, i.e.,

$$F_1(x) \geq F_2(x) \geq \dots \geq F_k(x) \quad \forall x.$$

The above condition is a consequence of Condition (1.1), but is not equivalent to it. Also, the approach of Brunk et al. is not Bayesian.

We shall first present a methodology for a Bayesian estimation of the individual hazard rate  $\lambda_j(x)$  unconstrained by (1.1).

## 2. Bayesian Estimation of a Single Hazard Rate Function

For estimating an individual hazard rate  $\lambda_j(t)$ , we use the following Bayesian procedure.

Let  $N_j(t)$  be the number of items exposed to the environment  $E_j$  at time  $t$ . Thus,  $N_j(0)$  is the number of items that are initially put on test in environment  $E_j$ .

We hypothesize that the failure times are governed by a time-dependent Poisson process with the probability of failure in  $[t, t+h]$  given by

$$N_j(t)\lambda_j(t)h + o(h),$$

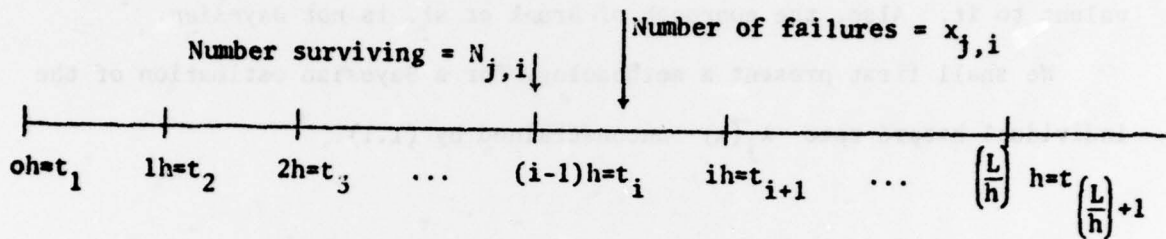
where  $\lambda_j(t)$  is the failure rate at time  $t$  under environment  $E_j$ .

For purposes of analysis, we divide the time interval  $[0, L]$  into intervals of length  $h > 0$ , where  $h$  is chosen to make  $L$  a multiple of  $h$ . For convenience, we denote  $(i-1)h$  by  $t_i$ ,  $N_j(t_i)$  by  $N_{j,i}$ , and  $\lambda_j(t_i)$  by  $\lambda_{j,i}$ , for  $i = 1, 2, \dots, (L/h)$ .



Let  $x_{j,i}$  denote the number of failures in  $[t_i, t_{i+h})$ ; if there are no withdrawals, removals, censoring, etc., then  $N_{j,i+1} = N_{j,i} - x_{j,i}$ .

It is helpful to clarify the above terms by the following diagram:



The joint distribution of  $x_{j,1}, x_{j,2}, \dots, x_{j,(L/h)+1}$  is given by the multivariate distribution

$$(N_{j1})! \prod_{i=1}^{(L/h)+1} (p_{j,i})^{x_{j,i}} / x_{j,i}!, \quad (2.1)$$

where  $p_{j,i}$  is the probability of failure of a specified unit in  $[t_i, t_{i+1})$ . with

$$p_{j,(L/h)+1} = 1 - \sum_{i=1}^{(L/h)} p_{j,i}, \quad \text{and} \quad x_{j,(L/h)+1} = N_{j,1} - \sum_{i=1}^{(L/h)} x_{j,i}.$$

Denoting  $p_{j,i} / \left[ 1 - \sum_{\ell=1}^{i-1} p_{j,\ell} \right]$  by  $p_{j,i}^*$ , Equation (2.1) can be expressed

[Wilks (1962), p. 151, Problem 6.13] as

$$\prod_{i=1}^{(L/h)} \frac{(N_{j,i})!}{(x_{j,i})! (N_{j,i} - x_{j,i})!} (p_{j,i}^*)^{x_{j,i}} (1 - p_{j,i}^*)^{N_{j,i} - x_{j,i}}. \quad (2.2)$$

The  $p_{j,i}^*$ 's can be interpreted as the *average failure rate* over the interval  $[t_i, t_{i+1})$ ; that is,  $p_{j,i}^*$  is the conditional probability that an item which survived to time  $t_i$  will fail by time  $t_{i+1}$ . In terms of the  $\lambda_{j,i}$ , we can write

$$\lambda_{j,i} h = p_{j,i}^* + o(h).$$

Equation (2.2) provides us with a starting point for a Bayesian analysis.

### 2.1. Prior Distributions for $p_{j,i}^*$

We shall confine the discussion in this section to the  $j^{\text{th}}$  environmental condition.

Suppose that the  $p_{j,i}^*$ ,  $i = 1, 2, \dots, (L/h)$ , are *a priori* independent beta random variables with marginal densities

$$f(p_{j,i}^*) = \frac{\Gamma(\alpha_{j,i} + \beta_{j,i})}{\Gamma(\alpha_{j,i})\Gamma(\beta_{j,i})} (p_{j,i}^*)^{\alpha_{j,i}-1} (1-p_{j,i}^*)^{\beta_{j,i}-1}.$$

It is of interest to note [cf. Lochner (1975)] that the above prior density leads us to a *generalized Dirichlet density* for the

$$p_j = (p_{j,1}, p_{j,2}, \dots, p_{j,(L/h)+1}).$$

Given the  $N_{j,i}$ 's and the  $x_{j,i}$ 's, the *posterior density* of  $(p_{j,1}^*, p_{j,2}^*, \dots, p_{j,(L/h)}^*)$  is

$$\prod_{i=1}^{(L/h)} \frac{\Gamma(\alpha_{j,i} + \beta_{j,i} + N_{j,i})}{\Gamma(\alpha_{j,i} + x_{j,i}) \Gamma(\beta_{j,i} + N_{j,i} - x_{j,i})} (p_{j,i}^*)^{\alpha_{j,i} + x_{j,i} - 1} (1 - p_{j,i}^*)^{\beta_{j,i} + N_{j,i} - x_{j,i} - 1}; \quad (2.3)$$

see DeGroot (1970) p. 160.

### 3. A Bayesian Estimation of Ordered Average Failure Rates.

In a Bayesian context, Condition (1.1) leads us to the requirement that for every fixed value of  $i$ ,

$$p_{j-1,i}^* \stackrel{st}{\geq} p_{j,i}^* \quad \text{for } j = 2, 3, \dots, k. \quad (3.1)$$

Thus, our prior distributions on  $p_{j-1,i}^*$  and  $p_{j,i}^*$  will have to be chosen such that

$$P[p_{j-1,i}^* \geq p] \geq P[p_{j,i}^* \geq p] \quad \text{for all } p \geq 0. \quad (3.2)$$

One way of achieving Condition (3.2) is to assume that the parameters of the prior distributions of  $p_{j-1,i}^*$  and  $p_{j,i}^*$  satisfy the following conditions for every fixed value of  $i$ :

$$\alpha_{j,i} \leq \alpha_{j-1,i}$$

and

$$\beta_{j,i} \geq \beta_{j-1,i} \quad \text{for } j = 2, 3, \dots, k. \quad (3.3)$$

For a proof of the above statement, see Appendix A.

In order to be assured that Condition (3.1) is also satisfied with respect to the posterior distributions of  $p_{j-1,i}^*$  and  $p_{j,i}^*$ , it is sufficient (see Appendix A) that, for every fixed value of  $i$

$$\alpha_{j,i} + x_{j,i} \leq \alpha_{j-1,i} + x_{j-1,i}$$

and

(3.4)

$$\beta_{j,i} + N_{j,i} - x_{j,i} \geq \beta_{j-1,i} + N_{j-1,i} - x_{j-1,i} \quad \text{for } j = 2, 3, \dots, k.$$

In order to make our Bayesian analysis more practical, we will have to reduce the number of prior parameters. One way of doing this is to assume that

$$\alpha_{j,i} = \alpha \quad \text{for all values of } i \text{ and } j,$$

and that

(3.5)

$$\beta_{j,i} = \beta_j \quad \text{for } i = 1, 2, \dots, (L/h).$$

Thus, the prior distribution of each  $p_{j,i}^*$ ,  $i = 1, 2, \dots, (L/h)$ , is a beta with parameters  $\alpha$  and  $\beta_j$ . This plus Conditions (3.3) and (3.4) lead us to the following remarks.

The first part of Condition (3.4) will be satisfied whenever  $x_{j,i} \leq x_{j-1,i}$  for  $j = 2, 3, \dots, k$ . That is, the number of failures in the interval  $[t_i, t_i+h)$  under environment  $E_j$  must not be greater than the number of failures in the same interval under environment  $E_{j-1}$ , for all values of  $j$ . Because Condition (3.4) will have to be satisfied for every fixed value of  $i$ ,  $i = 1, 2, \dots, (L/h)$ ,



a reasonable strategy is to have  $N_{j,1} < N_{j-1,1}$  for  $j = 2, 3, \dots, k$ . That is, we shall put more items on test (initially) under the more severe environment  $E_{j-1}$ , than under the environment  $E_j$ .

Since  $N_{j,i} = N_{j,i-1} - x_{j,i-1}$ ,  $i = 1, 2, \dots, (L/h)$ , the second part of Condition (3.4) can be written as

$$\beta_{j-1} + N_{j-1,i+1} \leq \beta_j + N_{j,i+1} \quad \text{for } j = 2, 3, \dots, k. \quad (3.6)$$

An interpretation of the above condition is that, for every fixed value of  $i$ , the number surviving at the start of the  $(i+1)^{\text{st}}$  interval plus the prior parameter  $\beta_{j-1}$  for the environment  $E_{j-1}$ , must be smaller than the corresponding sum for the environment  $E_j$ . Thus, whether the second part of Condition (3.4) is satisfied or not, depends not only on the number of failures in the  $i^{\text{th}}$  interval and the number surviving, but also on the values of the prior parameters  $\beta_j$  and  $\beta_{j-1}$ . Since the number of failures in a particular interval is a function of the severity of the environment and the number on test, and since  $\beta_{j-1}$  has to be less than or equal to  $\beta_j$  (see Conditions (3.5) and (3.6)), it is reasonable to have  $\beta_{j-1} < \beta_j$  whenever  $E_j < E_{j-1}$ . That is, the values of the prior parameters  $\beta_{j-1}$  and  $\beta_j$  are indicative of the relative severity of the environmental conditions  $E_{j-1}$  and  $E_j$ . Since  $E_1 > E_2 > \dots > E_k$ , we will choose the  $\beta_j$ 's such that  $\beta_1 < \beta_2 < \dots < \beta_k$ , and the values of the  $\beta_j$ 's will be indicative of the severity of the environmental conditions  $E_j$ ,  $j = 1, 2, \dots, k$ .

If the prior parameters  $\beta_j$ ,  $j = 1, 2, \dots, k$ , and the data from the accelerated life test,  $x_{j,i}$  and  $N_{j,i}$ ,  $i = 1, 2, \dots, (L/h)$ , are such that

Condition (3.4) is satisfied for every fixed value of  $i$ , then the stochastic ordering Condition (3.1) will be automatically satisfied with respect to the posterior distribution of  $(p_{j,1}^*, p_{j,2}^*, \dots, p_{j,(L/h)}^*)$ . If the above is not the case, then we will have to pool the adjacent violators using the pooling procedure described in the next section.

#### 4.0. The Pooling of Adjacent Violators.

The procedure for pooling adjacent violators described here is commonly used in *isotonic regression*; see Barlow, Bartholomew, Bremner and Brunk (1972). The pooling is between the violators of the assumed ordering; that is, whenever  $x_{j-1,i} < x_{j,i}$ , then  $x_{j-1,i}$  and  $x_{j,i}$  are pooled.

Consider the time interval  $[(i-1)h, ih]$ ; by Condition (3.4) we require

$$x_{1,i} \geq x_{2,i} \geq \dots \geq x_{j-1,i} \geq x_{j,i} \geq \dots \geq x_{k,i}$$

and

$$\begin{aligned} \beta_1 + N_{1,i} - x_{1,i} &\leq \beta_2 + N_{2,i} - x_{2,i} \leq \dots \leq \beta_{j-1} + N_{j-1,i} - x_{j-1,i} \\ &\leq \beta_j + N_{j,i} - x_{j,i} \leq \dots \leq \beta_k + N_{k,i} - x_{k,i}. \end{aligned}$$

If the reversal occurs, that is, if either

$$x_{j-1,i} < x_{j,i}$$

or if

$$\beta_{j-1} + N_{j-1,i} - x_{j-1,i} > \beta_j + N_{j,i} - x_{j,i},$$

then we pool the violators and replace them as shown below.

Replace both  $x_{j-1,i}$  and  $x_{j,i}$  by  $\frac{1}{2}(x_{j-1,i} + x_{j,i})$  and  $\beta_{j-1} + N_{j-1,i} - x_{j-1,i}$  and  $\beta_j + N_{j,i} - x_{j,i}$  by  $\frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i})$ .

We now test to see if the new sequence is properly ordered, i.e.,

$$x_{1,i} \geq x_{2,i} \geq \dots \geq \frac{1}{2}(x_{j-1,i} + x_{j,i}) = \frac{1}{2}(x_{j-1,i} + x_{j,i}) \geq$$

$$x_{j+1,i} \geq \dots \geq x_{k,i},$$

and

$$\begin{aligned} \beta_1 + N_{1,i} - x_{1,i} &\leq \dots \leq \frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i}) \\ &= \frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i}) \leq \dots \leq \beta_k + N_{k,i} - x_{k,i}. \end{aligned}$$

If a reversal exists in either of the new sequences, then we replace again by appropriate averages. Thus, if

$$\frac{1}{2}(x_{j-1,i} + x_{j,i}) = \frac{1}{2}(x_{j-1,i} + x_{j,i}) < x_{j+1,i},$$

or if

$$\frac{1}{2}(\beta_{j-1} + \beta_j + N_{j-1,i} + N_{j,i} - x_{j-1,i} - x_{j,i}) > \beta_{j+1} + N_{j+1,i} - x_{j+1,i},$$

then we replace each one of the three by the corresponding average

$$\frac{1}{3}(x_{j-1,i} + x_{j,i} + x_{j+1,i})$$



(in the first sequence) and

$$\frac{1}{3}(\beta_{j-1} + \beta_j + \beta_{j+1} + N_{j-1,i} + N_{j,i} + N_{j+1,i} - x_{j-1,i} - x_{j,i} - x_{j+1,i})$$

(in the second sequence).

We continue the above procedure until all reversals in the interval  $[(i-1)h, ih)$  are eliminated. We use this pooling scheme for each of the  $(L/h)$  time intervals to achieve the desired ordering.

#### 4.1. Some General Comments Regarding the Pooling Procedure

The following comments regarding the pooling of adjacent violators will be helpful.

1. An excessive amount of pooling occurs if the relationship specified by Condition (1.1) is incorrect, or if the environmental conditions  $E_1, E_2, \dots, E_k$  are too similar to each other.

2. If  $\lambda_j(0^+) \equiv 0$  for  $j = 1, 2, \dots, k$ , then it is reasonable to expect that some pooling will be necessary at the lower values of  $i$ ,  $i = 1, 2, \dots, (L/h)$ .

For many practical situations, it is reasonable to assume that the failure distributions  $F_j$ ,  $j = 1, 2, \dots, k, u$ , have increasing failure rate (see Barlow and Proschan (1975)). If we wish to make such an assumption, and incorporate it into our analysis, then, for each value of  $j$ , we must have in addition to Condition (3.1),

$$\lambda_{j,i-1} \leq \lambda_{j,i},$$

or that

$$p_{j,i-1}^{st} \geq p_{j,i}^{st}, \quad i = 1, 2, \dots, (L/h).$$

The above condition will further complicate our pooling procedure; we shall therefore *not* assume that the  $F_j$  are increasing failure rate.

# 5. The Posterior Distributions of the $p_{j,i}^*$ 's.

It is apparent from the discussion in Sections 3 and 4 that the posterior distributions of the  $p_{j,i}^*$ 's depend on the outcome of our pooling procedure. The posterior analysis is straightforward if no pooling is necessary, for then, the posterior distribution of  $(p_{j,1}^*, p_{j,2}^*, \dots, p_{j,(L/h)+1}^*)$  is simply

$$\prod_{i=1}^{(L/h)} \frac{\Gamma(\alpha + \beta_{j,i} + N_{j,i})}{\Gamma(\alpha + x_{j,i}) \Gamma(\beta_{j,i} + N_{j,i} - x_{j,i})} (p_{j,i}^*)^{\alpha + x_{j,i} - 1} (1 - p_{j,i}^*)^{\beta_{j,i} + N_{j,i} - x_{j,i} - 1}$$

for  $j = 1, 2, \dots, k$ .

Under the assumption of a squared error loss function, a Bayes estimator of  $p_{j,i}^*$ ,  $i = 1, 2, \dots, (L/h)$ ,  $j = 1, 2, \dots, k$ , is simply the posterior mean; that is

$$\hat{p}_{j,i}^* = \frac{\alpha + x_{j,i}}{(\alpha + \beta_{j,i} + N_{j,i})},$$

(see DeGroot (1970) p. 40).

If pooling is necessary, then some or all of the  $x_{j,i}$ 's and the  $(\beta_{j,i} + N_{j,i})$ 's will be replaced by their appropriately pooled averages. In any case, the general expression for  $\hat{p}_{j,i}^*$  will be of the form given above.

## 6. A Model for Extrapolations to Use Conditions Stress.

Our analysis leads us to an array of Bayes estimators of the average failure rates over intervals of length  $h$ , for each environmental condition. Because of our pooling strategy, the Bayes estimators will be correctly ordered. That is, for each fixed value of  $i$ ,

$$\hat{p}_{1,i}^* \geq \hat{p}_{2,i}^* \geq \dots \geq \hat{p}_{k,i}^*. \quad (6.1)$$

Given the  $\hat{p}_{j,i}^*$ 's,  $j = 1, 2, \dots, k$  and  $i = 1, 2, \dots, (L/h)$ , our goal is to predict  $p_{u,1}^*, p_{u,2}^*, \dots, p_{u,(L/h)}^*$ , the average failure rates over the time intervals  $[(i-1)h, ih)$ ,  $i = 1, 2, \dots, (L/h)$ , respectively, under the use conditions environment  $E_u$ .

In the absence of any knowledge about a relationship between the average failure rates and the values of the various stresses which constitute an environment, some form of an assumption is essential. This is particularly crucial if  $k$ , the number of distinct environmental conditions of interest, is small. If, however,  $k$  is large, then a relationship between the average failure rates and the stresses can be empirically obtained; this is what is often done in practice.

When  $k$  is small, we shall *postulate* the following simple but reasonable relationship between the average failure rate estimates.

For some  $k$  *unknown* constants  $w_0, w_1, \dots, w_{k-1}$ , we assume that for each value of  $i$ ,  $i = 1, 2, \dots, (L/h)$ ,

$$\hat{p}_{k,i}^* = w_0 + w_1 \hat{p}_{k-1,i}^* + w_2 \hat{p}_{k-2,i}^* + \dots + w_{k-1} \hat{p}_{1,i}^*. \quad (6.2)$$

The above relationship states that the average failure rate over a particular time interval under the environmental condition  $E_k$  is a weighted sum of the average failure rates over the same time interval under the conditions  $E_{k-1}, E_{k-2}, \dots, E_1$ . This is reminiscent of an *autoregressive process* of order  $k-1$  which has found useful applications in forecasting (see Box and Jenkins (1975)).

In order to make the above relationship more meaningful we shall require that the environmental conditions  $E_k < E_{k-1} < \dots < E_2 < E_1$  increase in magnitude of severity by the same fixed amount. For example, if  $E_j$  represents a single stress, say a voltage stress  $V_j$ , then we shall require that  $V_{j-1} - V_j = C$ , where  $C$  is some suitable constant,  $j = 1, 2, \dots, k$ .

Since Equation (6.2) holds for  $i = 1, 2, \dots, (L/h)$ , the least squares estimators of  $w_0, w_1, \dots, w_{k-1}$  can be obtained in a routine manner. These estimators are denoted by  $\hat{w}_0, \hat{w}_1, \dots, \hat{w}_{k-1}$ .

If the increase in the magnitude of the severity of the environment from  $E_u$  to  $E_k$  is the same as that from  $E_j$  to  $E_{j-1}$ ,  $j = 2, 3, \dots, k$ , then an estimator of the average failure rate under  $E_u$  is given by

$$\hat{p}_{u,i}^* = \hat{w}_0 + \hat{w}_1 \hat{p}_{k,i}^* + \dots + \hat{w}_{k-1} \hat{p}_{2,i}^* \quad (6.3)$$

for  $i = 1, 2, \dots, (L/h)$ , and  $\hat{p}_{u,(L/h)+1}^* = 1 - \sum_{i=1}^{(L/h)} \hat{p}_{u,i}^*$ .

If the increase in magnitude of the severity of the environment from  $E_u$  to  $E_k$  is two times that from  $E_j$  to  $E_{j-1}$ ,  $j = 2, 3, \dots, k$ , then we iterate



upon Equation (6.3) one more time to obtain the desired result. Thus, in principle, we can iterate upon Equation (6.3) as many times as is necessary, depending upon the separation between  $E_u$  and  $E_k$ .

By the definition of  $p_{u,i}^*$ , the probability of an item surviving to time  $t^*$  (assumed a multiple of  $h$ ) under environment  $E_u$  is

$$\bar{F}_u(t^*) = \prod_{i=1}^{t^*/h} (1 - p_{u,i}^*).$$

Thus, we have the following as an estimator of  $\bar{F}_u(t^*)$ :

$$\hat{\bar{F}}_u(t^*) = \prod_{i=1}^{t^*/h} (1 - \hat{p}_{u,i}^*),$$

where the  $\hat{p}_{u,i}^*$  are given by (6.3).

The properties of this estimator have not been studied. The estimation procedure has been proposed as a practical method for providing answers in the difficult area of accelerated testing.

## APPENDIX A.

In this appendix we determine sufficient conditions under which two beta random variables are stochastically ordered. The results may already be known; however, they are included for completeness.

The notation and terminology are taken from Barlow and Proschan (1975).

We first note that a gamma density  $f(x, \alpha) = \frac{x^{\alpha-1} e^{-x}}{\Gamma(\alpha)}$  is  $TP_2$  in  $(x, \alpha)$ . Therefore,  $\bar{F}(x, \alpha) = \int_x^\infty \frac{y^{\alpha-1}}{\Gamma(\alpha)} e^{-y} dy$  is  $TP_2$  in  $(x, \alpha)$  since  $\bar{F}(x, \alpha) = \int f(y, \alpha) H(x, y) dy$ , where  $H(x, y) = 1$  for  $y \geq x$  and  $H(x, y) = 0$  for  $y < x$ . (Note that  $\begin{vmatrix} H(x_1, y_1) & H(x_1, y_2) \\ H(x_2, y_1) & H(x_2, y_2) \end{vmatrix} \geq 0$  for  $x_1 < x_2$ ,  $y_1 < y_2$ . Thus,  $H(x, y)$  is  $TP_2$ , and so the composition of  $f, H$  is  $TP_2$ .)

Since  $\bar{F}(x, \alpha)$  is  $TP_2$ , then for  $x_1 < x_2$ ,  $\alpha_1 < \alpha_2$ ,  $\begin{vmatrix} \bar{F}(x_1, \alpha_1) & \bar{F}(x_1, \alpha_2) \\ \bar{F}(x_2, \alpha_1) & \bar{F}(x_2, \alpha_2) \end{vmatrix} \geq 0$ . Choose  $x_1 = 0$ . Then  $\begin{vmatrix} 1 & 1 \\ \bar{F}(x, \alpha_1) & \bar{F}(x, \alpha_2) \end{vmatrix} \geq 0$  for  $0 < x < \infty$ ,  $\alpha_1 < \alpha_2$ . Thus,  $\bar{F}(x, \alpha_2) \geq \bar{F}(x, \alpha_1)$  for  $\alpha_1 < \alpha_2$  and  $x > 0$ .

We have shown:

1. Lemma. Let  $X_\alpha$  be a gamma r.v. with shape parameter  $\alpha$ . Then  $X_\alpha$  is increasing stochastically in  $\alpha$ .

2. Proposition. Let  $Y_{\alpha_i}$  have density  $\frac{y^{\alpha_i-1}}{\Gamma(\alpha_i)} e^{-y}$ ,  $i = 1, 2$ , and  $Y_{\alpha_1}, Y_{\alpha_2}$  be independent. Then  $X_{\alpha_1, \alpha_2} \equiv \frac{Y_{\alpha_1}}{Y_{\alpha_1} + Y_{\alpha_2}}$  has beta density

$f(\alpha_1, \alpha_2, x) = \frac{\Gamma(\alpha_1 + \alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)} x^{\alpha_1-1} (1-x)^{\alpha_2-1}$  for  $0 \leq x \leq 1$ . (Hogg and Craig, 1970, p. 134).

3. Theorem. Let  $X_{\alpha_1, \alpha_2}$  have beta density  $f(\alpha_1, \alpha_2, x)$ . Then  $X_{\alpha_1, \alpha_2}$  is increasing stochastically in  $\alpha_1$  and decreasing stochastically in  $\alpha_2$ .

Proof. Write  $X_{\alpha_1, \alpha_2} = \frac{Y_{\alpha_1}}{Y_{\alpha_1} + Y_{\alpha_2}}$ ,  $X_{\alpha_1, \alpha'_2} = \frac{Y_{\alpha_1}}{Y_{\alpha_1} + Y_{\alpha'_2}}$ , where  $\alpha_2 < \alpha'_2$ .

For fixed  $Y_{\alpha_1}$ ,  $X_{\alpha_1, \alpha_2} \stackrel{st}{\geq} X_{\alpha_1, \alpha'_2}$ . By unconditioning  $X_{\alpha_1, \alpha_2} \stackrel{st}{>} X_{\alpha_1, \alpha'_2}$  for  $\alpha_2 < \alpha'_2$ .

By similar reasoning, we can show  $X_{\alpha_1, \alpha_2} \stackrel{st}{\leq} X_{\alpha'_1, \alpha_2}$  for  $\alpha_1 < \alpha'_1$ . ||



# REFERENCES

- [1] BARLOW, R. E., D. J. BARTHOLOMEW, J. M. BRENNER and H. D. BRUNK (1972). Statistical Inference Under Order Restrictions. Wiley, New York.
- [2] BARLOW, R. E. and F. PROSCHAN (1975). Statistical Theory of Reliability and Life Testing. Holt, Rinehart, and Winston, New York.
- [3] BOX, G. E. P. and G. M. JENKINS (1976). Time Series Analysis, Forecasting and Control. Holden-Day, San Francisco.
- [4] BRUNK, H. D., W. E. FRANCK, D. L. HANSON and R. V. HOGG (1966). Maximum likelihood estimation of the distributions of two stochastically ordered random variables. J. Amer. Statist. Assoc. 6 1067-1080.
- [5] DeGROOT, M. H. (1970). Optimal Statistical Decision. McGraw-Hill, New York.
- [6] LOCHNER, R. H. (1975). A generalized Dirichlet distribution in Bayesian life testing. J. Roy. Statist. Soc. B 37 103-113.
- [7] MANN, N. R., R. E. SCHAFER, and N. D. SINGPURWALLA (1974). Methods for Statistical Analysis of Reliability and Life Data. Wiley, New York.
- [8] PATHAK, P. K., W. J. ZIMMER, and R. E. WILLIAMS (1977). Estimates of parameters in acceleration models. Technical Report No. 334. The University of New Mexico, Albuquerque, New Mexico.
- [9] SHAKED, M., W. J. ZIMMER, and C. A. BALL (1977). A nonparametric approach to accelerated life testing. Technical Report No. 340. The University of New Mexico, Albuquerque, New Mexico.
- [10] WILKS, S. S. (1963). Mathematical Statistics. Wiley, New York.